MA 234, Spring 2016

Assignment 5

This homework is due Friday Feb 26.

There are total 45 points in this assignment. 40 points is considered 100%. If you go over 40 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should exhibit your work and contain full proofs. Bare answers will not earn you much.

This assignment covers Sections 2.4–3.2 of Textbook.

1. Branches of functions

(1) [5pt] Let $f_1(z)$ be the principal square root function and $f_2(z)$ be the complementing branch of square root, $f_2(z) = -f_1(z)$. Use polar coordinates to find and sketch image of
  - (a) quadrant II ($x < 0$, $y > 0$) under the mapping $w = f_1(z)$,
  - (b) quadrant II ($x < 0$, $y > 0$) under the mapping $w = f_2(z)$,
  - (c) the right half-plane $\mathrm{Re}(z) > 0$ under the mapping $w = f_1(z)$,
  - (d) the right half-plane $\mathrm{Re}(z) > 0$ under the mapping $w = f_2(z)$.

(2) [5pt] Describe and sketch Riemann surface for $z^{\frac{1}{3}}$. (What sheets does it consist of? How are they attached to each other?).

2. Mapping $w = \frac{1}{z}$

(3) [10pt] Find the images of the mapping $w = 1/z$ in each case, and sketch the mapping.
  - (a) The horizontal line $\{(x, y) : y = \frac{1}{4}\}$.
  - (b) The vertical line $\mathrm{Re}(z) = -3$.
  - (c) The circle $C_{\frac{1}{2}}(-\frac{i}{2}) = \{z : |z + \frac{i}{2}| = \frac{1}{2}\}$.
  - (d) The circle $C_1(-2) = \{z : |z + 2| = 1\}$.
  - (e) The line $2x + 2y = 1$.

(4) [5pt]
  - (a) Show that transformation $w = 1/z$ maps the vertical strip given by $0 < x < \frac{1}{2}$ onto the region in the right half-plane $\mathrm{Re}(w) > 0$ that lies outside the disk $D_1(1) = \{w : |w - 1| = 1\}$.
  - (b) Find the image of the disk $D_{\frac{4}{3}}(\frac{-2i}{3}) = \{z : |z + \frac{2i}{3}| < \frac{4}{3}\}$ under $f(z) = 1/z$.

3. Derivative

(5) [5pt] Prove the following directly by computing the limit in the definition of the derivative.
  - (a) $(z^3)' = 3z^2$.
  - (b) $\left(\frac{1}{z}\right)' = \frac{-1}{z^2}$.

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(6) [5pt] Find the derivative of the following functions using rules of differentiation.

(a) $(z^2 - iz + 9)^5$. (Simplifying the answer is not necessary.)
(b) $\frac{2z+1}{z+2}$.
(c) $(z^2 + (1-2i)z + 1)(z^2 + 3z^2 + 5i)$.

(7) [10pt] Use the Cauchy–Riemann conditions to determine where the following functions are differentiable and evaluate the derivatives at those points where they do exist.

(a) $f(z) = f(x, y) = \frac{y+ix}{x^2+y^2}$.
(b) $f(z) = -2(xy + x) + i(x^2 - 2y - y^2)$.
(c) $f(z) = x^3 + i(1 - y^3)$.
(d) $f(z) = x^2 + y^2 + 2ixy$.
(e) $f(z) = e^y \cos x + ie^y \sin x$
(f) $f(z) = \cosh x \sin y - i \sinh x \cos y$